

ON THE REGULARITY OF WEAK SOLUTIONS OF THE 3D NAVIER-STOKES EQUATIONS IN $B_{\infty,\infty}^{-1}$

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ABSTRACT. We show that if a Leray-Hopf solution u to the 3D Navier-Stokes equation belongs to $C((0, T]; B_{\infty,\infty}^{-1})$ or its jumps in the $B_{\infty,\infty}^{-1}$ -norm do not exceed a constant multiple of viscosity, then u is regular on $(0, T]$. Our method uses frequency local estimates on the nonlinear term, and yields an extension of the classical Ladyzhenskaya-Prodi-Serrin criterion.

1. INTRODUCTION

We study the 3D incompressible Navier-Stokes equations (NSE)

$$(1) \quad \begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla p = 0, & x \in \mathbb{R}^3, t \geq 0, \\ \nabla \cdot u = 0, \\ u(0) = u_0, \end{cases}$$

where $u(x, t)$, the velocity, and $p(x, t)$, the pressure, are unknowns, $u_0 \in L^2(\mathbb{R}^3)$ is the initial condition, and $\nu > 0$ is the kinematic viscosity coefficient of the fluid.

In 1934 Leray [9] proved that for every divergence-free initial data $u_0 \in L^2(\mathbb{R}^3)$ there exists a weak solution to the system (1) on $[0, \infty)$ with $u(0) = u_0$. Moreover, one can find a weak solution satisfying the energy inequality (see Section 2), called the Leray-Hopf solution.

A famous open question, equivalent to one of the millennium problems, is whether all Leray-Hopf solutions to (1) on $[0, \infty)$ with smooth initial data are regular. A weak solution $u(t)$ is called regular if $\|u(t)\|_{H^1}$ is continuous. Even though the regularity problem is far from been solved, numerous regularity criteria were proved since the work of Leray. The first result of this kind is due to Leray in the \mathbb{R}^3 case (see also Ladyzhenskaya, Serin, and Prodi [10, 12, 11] in the case of a bounded domain). It states that every Leray-Hopf solution u to (1) with $u \in L^r((0, T); L^s)$ is regular on $(0, T]$ provided $2/r + 3/s = 1$, $s \in (3, \infty]$. The limit case $r = \infty$ appeared to

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be more challenging. First, Leray [10] showed that Leray-Hopf solutions in $L^\infty((0, T); L^p)$ are regular provided $p > 3$. Later von Wahl [13] and Giga [7] increased the regularity space to $C((0, T); L^3)$, and, finally, Esperanza, Seregin, and Šverák [5] increased it further to $L^\infty((0, T); L^3)$.

Due to this remarkable result, if a solution loses regularity, then its L^3 -norm blows up. The space L^3 , as well as all the other critical spaces for the 3D NSE (see Cannone [1]), is continuously embedded in the Besov space $B_{\infty, \infty}^{-1}$. Hence, a natural question is whether $B_{\infty, \infty}^{-1}$ -norm has to blow up if a solution loses regularity. In this paper we prove a slightly weaker statement. Namely, if a Leray-Hopf solution u to (1) is continuous in $B_{\infty, \infty}^{-1}$ on the interval $(0, T]$, then u is regular on $(0, T]$. In fact we will prove that regularity of the solution is guaranteed even if u is discontinuous but all its jumps are smaller than a constant proportional to the viscosity (see Theorem 3.1). Thus, discontinuities of a limited size are allowed. As a consequence, we conclude that Leray-Hopf solutions with $B_{\infty, \infty}^{-1}$ -norm bounded by $c\nu$ are regular. This last result will also be shown in the "almost everywhere" version (see Corollary 3.3).

A related direction in the study of the 3D NSE concerns solutions with small initial data. The best present result due to Koch and Tataru [8] states that if the initial data is small in BMO^{-1} , then there exists a global in time regular solution of the 3D NSE with this initial data. Note that the space BMO^{-1} is continuously embedded in $B_{\infty, \infty}^{-1}$. The small initial data theorem in this larger space is unknown, but would be desirable to obtain.

We carry out our arguments in the case of \mathbb{R}^3 , but they apply to the periodic case as well.

2. PRELIMINARIES

2.1. Besov spaces. We will use the notation $\lambda_q = 2^q$ (in some inverse length units). Let B_r denote the ball centered at 0 of radius r in \mathbb{R}^3 . Let us fix a nonnegative radial function $\chi \in C_0^\infty(B_2)$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1$. We further define

$$(2) \quad \varphi(\xi) = \chi(\lambda_1^{-1}\xi) - \chi(\xi).$$

and

$$(3) \quad \varphi_q(\xi) = \varphi(\lambda_q^{-1}\xi).$$

For a tempered distribution vector field u let us denote

$$\begin{aligned} u_q &= \mathcal{F}^{-1}(\varphi_q) * u, \quad \text{for } q > -1, \\ u_{-1} &= \mathcal{F}^{-1}(\chi) * u, \end{aligned}$$

where \mathcal{F} denotes the Fourier transform. So, we have

$$u = \sum_{q=-1}^{\infty} u_q$$

in the sense of distributions. We also use the following notation

$$u_{\leq q} = \sum_{p=-1}^q u_p.$$

Let us recall the definition of Besov spaces. A tempered distribution u belongs to $B_{p,\infty}^s$ for some $s \in \mathbb{R}$ and $p \geq 1$ iff

$$\|u\|_{B_{p,\infty}^s} = \sup_{q \geq -1} \lambda_q^s \|u_q\|_p < \infty.$$

2.2. Weak solutions of the 3D NSE. In this section we recall some of the classical definitions and results on weak solutions of the NSE.

Definition 2.1. A weak solution of (1) on $[0, T]$ (or $[0, \infty)$ if $T = \infty$) is a function $u : [0, T] \rightarrow L^2(\mathbb{R}^3)$ in the class

$$u \in C_w([0, T]; L^2(\mathbb{R}^3)) \cap L_{\text{loc}}^2([0, T]; H^1(\mathbb{R}^3)),$$

satisfying

$$(4) \quad (u(t), \varphi(t)) + \int_0^t \{-(u, \partial_t \varphi) + \nu(\nabla u, \nabla \varphi) + (u \cdot \nabla u, \varphi)\} ds = (u_0, \varphi(0)),$$

for all $t \in [0, T]$ and all test functions $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^3)$ with $\nabla \cdot \varphi = 0$. Here (\cdot, \cdot) stands for the L^2 -inner product.

Theorem 2.2 (Leray). *For every $u_0 \in L^2(\mathbb{R}^3)$, there exists a weak solution of (1) on $[0, \infty)$ satisfying the following energy inequality*

$$(5) \quad \|u(t)\|_2^2 + 2\nu \int_{t_0}^t \|\nabla u(s)\|_2^2 ds \leq \|u(t_0)\|_2^2,$$

for almost all $t_0 \in [0, \infty)$ including $t_0 = 0$, and all $t \in [t_0, \infty)$.

Definition 2.3. A Leray-Hopf solution of (1) on $[0, T]$ is a weak solution on $[0, T]$ satisfying the energy inequality (5) for almost all $t_0 \in (0, T)$ and all $t \in [t_0, T]$.

A weak solution $u(t)$ of (1) on a time interval I is called regular if $\|u(t)\|_{H^1}$ is continuous on I . The following is a well-known fact concerning Leray-Hopf solutions. Note that weak solutions are not known to satisfy this property.

Theorem 2.4 (Leray). *Let $u(t)$ be a Leray-Hopf solution of (1) on $[0, T]$. If for every interval of regularity $(\alpha, \beta) \subset (0, T)$*

$$\limsup_{t \rightarrow \beta-} \|u(t)\|_{H^s} < \infty,$$

for some $s > 1/2$, then $u(t)$ is regular on $(0, T]$.

3. MAIN RESULT

In this section we state and prove our main result. Its proof is a consequence of the limiting case of other regularity criteria in the subcritical range of integrability parameter. We will discuss how those criteria extend some of the classical and newly found results on regularity.

Theorem 3.1. *Let $u(t)$ be a Leray-Hopf solution of (1) on $[0, T]$. If $u(t)$ satisfies*

$$(6) \quad \sup_{t \in (0, T]} \limsup_{t_0 \rightarrow t-} \|u(t) - u(t_0)\|_{B_{\infty, \infty}^{-1}} < c\nu,$$

where $c > 0$ is some absolute constant, then $u(t)$ is regular on $(0, T]$.

In particular, if

$$u \in C((0, T]; B_{\infty, \infty}^{-1}),$$

then $u(t)$ is regular on $(0, T]$.

The following criterion will be a crucial ingredient in the proof of Theorem 3.1.

Lemma 3.2. *Let $u(t)$ be a Leray-Hopf solution of (1) on $[0, T]$. There is a constant $c > 0$ such that if $u(t)$ satisfies*

$$(7) \quad \limsup_{q \rightarrow \infty} \sup_{t \in (0, T)} \lambda_q^{-1} \|u_q(t)\|_{\infty} < c\nu,$$

then $u(t)$ is regular on $(0, T]$.

Proof. Let $(\alpha, \beta) \subset (0, T]$ be an interval of regularity. Let us fix $\epsilon \in (0, 1)$ and use the weak formulation of the NSE with the test-function $\lambda_q^{1+\epsilon}(u_q)_q$. Then on (α, β) we obtain

$$(8) \quad \frac{1}{2} \frac{d}{dt} \lambda_q^{1+\epsilon} \|u_q\|_2^2 + \nu \lambda_q^{3+\epsilon} \|u_q\|_2^2 \leq \lambda_q^{1+\epsilon} \int \text{Tr}[(u \otimes u)_q \cdot \nabla u_q] dx.$$

We can write

$$(u \otimes u)_q = r_q(u, u) + u_q \otimes u + u \otimes u_q,$$

for all $q > -1$, where

$$r_q(u, u)(x) = \int_{\mathbb{R}^3} \mathcal{F}^{-1}(\varphi_q)(y) (u(x-y) - u(x)) \otimes (u(x-y) - u(x)) dy.$$

After proper cancelations we arrive at

$$\int_{\mathbb{R}^3} \text{Tr}[(u \otimes u)_q \cdot \nabla u_q] dx = \int_{\mathbb{R}^3} r_q(u, u) \cdot \nabla u_q dx - \int_{\mathbb{R}^3} u_q \cdot \nabla u_{\leq q+1} \cdot u_q dx.$$

Let us now estimate the first term using Hölder and Bernstein's inequalities

$$\int_{\mathbb{R}^3} r_q(u, u) \cdot \nabla u_q dx \lesssim \|r_q(u, u)\|_{3/2} \lambda_q \|u_q\|_3.$$

Using Littlewood-Paley decomposition we obtain as in [4]

$$\begin{aligned} \|r_q(u, u)\|_{3/2} &\lesssim \int_{\mathbb{R}^3} |\mathcal{F}^{-1}(\varphi_q)(y)| \|u(\cdot - y) - u(\cdot)\|_3^2 dy \\ &\lesssim \int_{\mathbb{R}^3} |\mathcal{F}^{-1}(\varphi_q)(y)| \left(\sum_{p=-1}^q |y|^2 \lambda_p^2 \|u_p\|_3^2 + \sum_{p=q+1}^{\infty} \|u_p\|_3^2 \right) dy \\ &\lesssim \sum_{p=-1}^q \lambda_q^{-2} \lambda_p^2 \|u_p\|_3^2 + \sum_{p=q+1}^{\infty} \|u_p\|_3^2. \end{aligned}$$

Analogously,

$$\int_{\mathbb{R}^3} u_q \cdot \nabla u_{\leq q+1} \cdot u_q dx \lesssim \|u_q\|_3^2 \sum_{p=-1}^{q+1} \lambda_p \|u_p\|_3.$$

Thus, up to a constant multiple independent of u , the nonlinear term in (8) is bounded by

$$\lambda_q^\epsilon \|u_q\|_3 \sum_{p=-1}^q \lambda_p^2 \|u_p\|_3^2 + \lambda_q^{2+\epsilon} \|u_q\|_3 \sum_{p=q+1}^{\infty} \|u_p\|_3^2 + \lambda_q^{2+\epsilon} \|u_q\|_3^2 \sum_{p=-1}^{q+1} \lambda_p \|u_p\|_3.$$

Let us now fix $Q \in \mathbb{N}$ and sum over $q \geq Q$ in (8) on (α, β) obtaining

$$(9) \quad \frac{1}{2} \frac{d}{dt} \sum_{q \geq Q} \lambda_q^{1+\epsilon} \|u_q\|_2^2 + \nu \sum_{q=Q}^{\infty} \lambda_q^{3+\epsilon} \|u_q\|_2^2 \lesssim I + II + III,$$

where

$$\begin{aligned} I &= \sum_{q=Q}^{\infty} \lambda_q^\epsilon \|u_q\|_3 \sum_{p=-1}^q \lambda_p^2 \|u_p\|_3^2, \\ II &= \sum_{q=Q}^{\infty} \lambda_q^{2+\epsilon} \|u_q\|_3 \sum_{p=q+1}^{\infty} \|u_p\|_3^2, \\ III &= \sum_{q=Q}^{\infty} \lambda_q^{1+\epsilon} \|u_q\|_3^2 \sum_{p=-1}^{q+1} \lambda_p \|u_p\|_3. \end{aligned}$$

We will show that the right hand side of (9) obeys the following estimate

$$(10) \quad I + II + III \lesssim \sum_{q=Q}^{\infty} \lambda_q^{2+\epsilon} \|u_q\|_3^3 + R(Q),$$

where

$$R(Q) = \sup_{t \in (0, T)} \sum_{q=-1}^{Q-1} \lambda_q^{2+\epsilon} \|u_q\|_3^3.$$

Once this is proved, the argument goes as follows. By interpolation, $\|u_q\|_3^3 \leq \|u_q\|_2^2 \|u_q\|_{\infty}$. So,

$$\frac{1}{2} \frac{d}{dt} \sum_{q=Q}^{\infty} \lambda_q^{1+\epsilon} \|u_q\|_2^2 + \nu \sum_{q=Q}^{\infty} \lambda_q^{3+\epsilon} \|u_q\|_2^2 \leq C \sum_{q=Q}^{\infty} \lambda_q^{3+\epsilon} \|u_q\|_2^2 (\lambda_q^{-1} \|u_q\|_{\infty}) + CR(Q),$$

where $C > 0$ is an absolute constant independent of u . Let $c = \nu/C$. Choosing Q so that $\lambda_q^{-1} \|u_q(t)\|_{\infty} < c$ for all $q \geq Q$ and all $t \in (0, T)$, we obtain

$$\frac{1}{2} \frac{d}{dt} \sum_{q=Q}^{\infty} \lambda_q^{1+\epsilon} \|u_q\|_2^2 \leq CR(Q),$$

for any $t \in (\alpha, \beta)$. Since $u(t)$ is bounded in L^2 on $(0, T]$, it follows that $R(Q) < \infty$. Hence the $H^{1/2+\epsilon}$ -norm of $u(t)$ is bounded on (α, β) . Therefore $u(t)$ is regular on $(0, T]$ due to Theorem 2.4.

Let us now prove (10). Indeed, using Young and Jensen's inequalities we obtain

$$\begin{aligned} I &= \sum_{q=Q}^{\infty} \lambda_q^{\epsilon} \|u_q\|_3 \sum_{p=-1}^q \lambda_p^2 \|u_p\|_3^2 \\ &= \sum_{q=Q}^{\infty} \lambda_q^{\frac{2}{3}+\frac{\epsilon}{3}} \|u_q\|_3 \sum_{p=-1}^q (\lambda_p \lambda_q^{-1})^{\frac{2}{3}-\frac{2\epsilon}{3}} \lambda_p^{\frac{4}{3}+\frac{2\epsilon}{3}} \|u_p\|_3^2 \\ &\lesssim \sum_{q=Q}^{\infty} \lambda_q^{2+\epsilon} \|u_q\|_3^3 + \sum_{q=Q}^{\infty} \sum_{p=-1}^q (\lambda_p \lambda_q^{-1})^{\frac{2}{3}-\frac{2\epsilon}{3}} \lambda_p^{2+\epsilon} \|u_p\|_3^3 \\ &\lesssim \sum_{q=Q}^{\infty} \lambda_q^{2+\epsilon} \|u_q\|_3^3 + R(Q). \end{aligned}$$

Similarly,

$$\begin{aligned}
II &= \sum_{q=Q}^{\infty} \lambda_q^{2+\epsilon} \|u_q\|_3 \sum_{p=q+1}^{\infty} \|u_p\|_3^2 \\
&= \sum_{q=Q}^{\infty} \lambda_q^{\frac{2}{3}+\frac{\epsilon}{3}} \|u_q\|_3 \sum_{p=q+1}^{\infty} (\lambda_q \lambda_p^{-1})^{\frac{4}{3}+\frac{2\epsilon}{3}} \lambda_p^{\frac{4}{3}+\frac{2\epsilon}{3}} \|u_p\|_3^2 \\
&\lesssim \sum_{q=Q}^{\infty} \lambda_q^{2+\epsilon} \|u_q\|_3^3 + \sum_{q=Q}^{\infty} \sum_{p=q+1}^{\infty} (\lambda_q \lambda_p^{-1})^{\frac{4}{3}+\frac{2\epsilon}{3}} \lambda_p^{2+\epsilon} \|u_p\|_3^3 \\
&\lesssim \sum_{q=Q}^{\infty} \lambda_q^{2+\epsilon} \|u_q\|_3^3.
\end{aligned}$$

And finally,

$$\begin{aligned}
III &= \sum_{q=Q}^{\infty} \lambda_q^{1+\epsilon} \|u_q\|_3^2 \sum_{p=-1}^{q+1} \lambda_p \|u_p\|_3 \\
&= \sum_{q=Q}^{\infty} \lambda_q^{\frac{4}{3}+\frac{2\epsilon}{3}} \|u_q\|_3^2 \sum_{p=-1}^{q+1} (\lambda_p \lambda_q^{-1})^{\frac{1}{3}-\frac{\epsilon}{3}} \lambda_p^{\frac{2}{3}+\frac{\epsilon}{3}} \|u_p\|_3 \\
&\lesssim \sum_{q=Q}^{\infty} \lambda_q^{2+\epsilon} \|u_q\|_3^3 + \sum_{q=Q}^{\infty} \sum_{p=-1}^{q+1} (\lambda_p \lambda_q^{-1})^{\frac{1}{3}-\frac{\epsilon}{3}} \lambda_p^{2+\epsilon} \|u_p\|_3^3 \\
&\lesssim \sum_{q=Q}^{\infty} \lambda_q^{2+\epsilon} \|u_q\|_3^3 + R(Q).
\end{aligned}$$

□

It is worth mentioning that this lemma immediately yields the following corollary. For mild solutions in the sense of Kato a similar result was proved in [2].

Corollary 3.3. *Let $u(t)$ be a Leray-Hopf solution of (1) on $[0, T]$. There is a constant $c > 0$ such that if $u(t)$ satisfies*

$$\|u\|_{L^\infty((0,T);B_{\infty,\infty}^{-1})} < c\nu,$$

then $u(t)$ is regular on $(0, T]$.

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Take any interval of regularity $(\alpha, \beta) \subset (0, T]$. Since $u(t)$ is a Leray-Hopf solution, $u(t) \in C^\infty(\mathbb{R}^3)$ for all $t \in (\alpha, \beta)$. Hence

$$\phi(t) := \limsup_{q \rightarrow \infty} \lambda_q^{-1} \|u_q(t)\|_\infty = 0, \quad \forall t \in (\alpha, \beta).$$

Then, thanks to (6), $\phi(\beta) < c\nu$. This together with (6) implies that there exists $t_0 \in (\alpha, \beta)$, such that

$$\limsup_{q \rightarrow \infty} \sup_{s \in (t_0, \beta)} \lambda_q^{-1} \|u_q(s)\|_\infty < 2c\nu.$$

Thus, if c is chosen to be half of what is in Lemma 3.2, then that lemma implies regularity of $u(t)$ on $(\alpha, \beta]$, i.e., continuity of $\|u(t)\|_{H^1}$ on $(\alpha, \beta]$. Hence $u(t)$ is regular on $(0, T]$ in view of Theorem 2.4. \square

4. EXTENSION OF THE LADYZHENSKAYA-PRODI-SERIN CRITERIA

We conclude with the following regularity criterion in the subcritical range of integrability exponent.

Lemma 4.1. *Let $u(t)$ be a Leray-Hopf solution of (1) on $[0, T]$. There is a constant $c > 0$ such that if $u(t)$ satisfies*

(11)

$$\sup_{t \in (0, T]} \lim_{t_0 \rightarrow t-} \limsup_{q \rightarrow \infty} \int_{t_0}^t \left(\lambda_q^{\frac{2}{r}-1} \|u_q(s)\|_\infty \right)^r ds < \nu^{r-1} c^r \left(\frac{r}{r-1} \right)^{r-1},$$

for some $r \in (2, \infty)$, then $u(t)$ is regular on $(0, T]$.

Proof. Let $(\alpha, \beta) \subset (0, T]$ be an interval of regularity. We use the weak formulation of the NSE with the test-function $\lambda_q^{2r-2}(u_q)_q$. Then on (α, β) we obtain

$$(12) \quad \frac{1}{2} \frac{d}{dt} \lambda_q^{2r-2} \|u_q\|_2^2 \leq -\nu \lambda_q^{2r} \|u_q\|_2^2 + \lambda_q^{2r-2} \int_{\mathbb{R}^3} \text{Tr}[(u \otimes u)_q \cdot \nabla u_q] dx.$$

As in the proof of Lemma 3.2, we have the following bound on the nonlinear term

$$\int_{\mathbb{R}^3} \text{Tr}[(u \otimes u)_q \cdot \nabla u_q] dx \leq C_1 \sum_{p=-1}^{\infty} \lambda_{|q-p|}^{-\frac{2}{3}} \lambda_p \|u_p\|_3^3,$$

where $C_1 > 0$ is an absolute constant. Thanks to Young and Jensen's inequalities,

$$\begin{aligned} \lambda_q^{2r-2} \|u_q\|_2^{2r-2} C_1 \sum_{p=-1}^{\infty} \lambda_{|q-p|}^{-\frac{2}{3}} \lambda_p \|u_p\|_3^3 &\leq \nu \lambda_q^{2r} \|u_q\|_2^{2r} \\ &\quad + (rA)^{-1} \sum_{p=-1}^{\infty} \lambda_{|q-p|}^{-\frac{2}{3}} \lambda_p^r \|u_p\|_3^{3r}, \end{aligned}$$

where

$$A = \nu^{r-1} C^{-r} \left(\frac{r}{r-1} \right)^{r-1},$$

and $C > 0$ is an absolute constant independent of u . Therefore on (α, β) we have

(13)

$$\begin{aligned} \frac{1}{2r} \frac{d}{dt} \lambda_q^{2r-2} \|u_q\|_2^{2r} &\leq -\nu \lambda_q^{2r} \|u_q\|_2^{2r} + \lambda_q^{2r-2} \|u_q\|_2^{2r-2} C_1 \sum_{p=-1}^{\infty} \lambda_{|q-p|}^{-\frac{2}{3}} \lambda_p \|u_p\|_3^3 \\ &\leq (rA)^{-1} \sum_{p=Q+1}^{\infty} \lambda_{|q-p|}^{-\frac{2}{3}} \lambda_p^r \|u_p\|_3^{3r} + R(Q), \end{aligned}$$

where

$$R(Q) = (rA)^{-1} \sup_{t \in (0, T)} \sum_{p=-1}^Q \lambda_p^r \|u_p\|_3^{3r},$$

$r \in (2, \infty)$, and $Q \in \mathbb{N}$. Note that since $u(t)$ is bounded in L^2 on $(0, T]$, we have $R(Q) < \infty$.

Now assume that (11) holds for $c = (C\sqrt{24})^{-1}$. Then there exist $t_0 \in (\alpha, \beta)$ and Q large enough, such that

$$\sup_{q > Q} \int_{t_0}^{\beta} (\lambda_q^{\frac{2}{r}-1} \|u_q(s)\|_{\infty})^r ds \leq \frac{1}{24} A.$$

Let $t \in (t_0, \beta)$. Integrating (13) we obtain

$$\begin{aligned} (14) \quad \sup_{s \in (t_0, t)} \lambda_q^{2r-2} \|u_q(s)\|_2^{2r} - \lambda_q^{2r-2} \|u_q(t_0)\|_2^{2r} \\ \leq 2A^{-1} \int_{t_0}^t \sum_{p=Q+1}^{\infty} \lambda_{|q-p|}^{-\frac{2}{3}} \lambda_p^r \|u_p(s)\|_3^{3r} ds + R(Q)(t - t_0). \end{aligned}$$

Thanks to Levi's convergence theorem we have

$$\begin{aligned} \int_{t_0}^t \sum_{p=Q+1}^{\infty} \lambda_{|q-p|}^{-\frac{2}{3}} \lambda_p^r \|u_p(s)\|_3^{3r} ds \\ \leq \sup_{p > Q} \left\{ \sup_{s \in (t_0, t)} \lambda_p^{2r-2} \|u_p(s)\|_2^{2r} \int_{t_0}^t (\lambda_p^{\frac{2}{r}-1} \|u_p(s)\|_{\infty})^r ds \right\} \sum_{p=1}^{\infty} \lambda_{|q-p|}^{-\frac{2}{3}} \\ \leq 6 \sup_{p > Q} \left\{ \sup_{s \in (t_0, t)} \lambda_p^{2r-2} \|u_p(s)\|_2^{2r} \right\} \sup_{p > Q} \int_{t_0}^t (\lambda_p^{\frac{2}{r}-1} \|u_p(s)\|_{\infty})^r ds \\ \leq \frac{1}{4} A \sup_{p > Q} \sup_{s \in (t_0, t)} \lambda_p^{2r-2} \|u_p(s)\|_2^{2r}. \end{aligned}$$

Hence, taking the supremum of (14) over $q > Q$, we obtain

$$\frac{1}{2} \sup_{q > Q} \sup_{s \in (t_0, t)} \lambda_q^{2r-2} \|u_q(s)\|_2^{2r} \leq \sup_{q > Q} \lambda_q^{2r-2} \|u_q(t_0)\|_2^{2r} + R(Q)(t - t_0).$$

Letting $t \rightarrow \beta$ we now have

$$\sup_{q>Q} \sup_{s \in (t_0, \beta)} \lambda_q^{1-\frac{1}{r}} \|u_q(s)\|_2 < \infty.$$

Thus, since $r > 2$, the solution $u(t)$ is bounded in H^s on (t_0, β) for some $s > 1/2$, which implies regularity in view of Theorem 2.4. \square

Let us now see how this lemma implies an extension of the classical Ladyzhenskaya-Serlin-Prodi condition in the subcritical range of integrability exponent. Indeed, by Bernstein inequalities and Littlewood-Paley Theorem we have the inclusions

$$B_{s,\infty}^{\frac{3}{r}+\frac{2}{r}-1} \subset B_{\infty,\infty}^{\frac{2}{r}-1},$$

$$L^s \subset B_{s,\infty}^0,$$

for all $s \geq 1$. On the other hand, as an easy consequence of Lemma 4.1 we have the following theorem.

Theorem 4.2. *Let $u(t)$ be a Leray-Hopf solution of (1) on $[0, T]$ satisfying*

$$(15) \quad u \in L^r((0, T); B_{\infty,\infty}^{\frac{2}{r}-1}),$$

for some $r \in (2, \infty)$. Then $u(t)$ is regular on $(0, T]$.

Let us also note that for a negative smoothness parameter as in (15), the homogeneous version of the Besov space is smaller than the nonhomogeneous one, i.e.,

$$\dot{B}_{\infty,\infty}^{\frac{2}{r}-1} \subset B_{\infty,\infty}^{\frac{2}{r}-1}, \quad \text{for all } r > 2.$$

In view of this fact, Theorem 4.2 extends the corresponding result obtained recently by Q. Chen and Z. Zhang in [3].

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